Introduction

Chemonics is committed to facilitating high-quality early childhood, primary and secondary numeracy programs in low-resource environments. Our commitment acknowledges that research demonstrating children's mathematics performance in early grades is a strong predictor of overall academic success in higher grades, particularly for mathematics and reading (Watts et al., 2014; Bodovski and Young, 2012; Duncan and Magnuson, 2011; Duncan et al., 2007; Purpura et al., 2011). Without strong foundations in numeracy, children are likely to repeat a grade (Duncan and Magnuson, 2011).

Chemonics is committed to improving numeracy instruction with evidence-based principles, and developing and disseminating processes and tools to operationalize high-quality numeracy instruction within education systems. The instructional models and teaching techniques presented in this toolkit provide responsive, inclusive instruction through practical application of core elements of Universal Design for Learning. Specifically, the toolkit offers methods to support and include all learners in the classroom that give more time and practice with multiple means of engagement, representation, actions and expressions (Hayes, Turnbull, and Moran, 2018). The toolkit principles are complementary to frameworks and guidelines provided in USAID's Reading MATTERS framework and Global Partnership for Education's Global Proficiency Framework: Reading and Mathematics.

This toolkit is a living document and presents our current perspectives on how to best serve local partners in the design of efficient and effective numeracy programs for young learners. In addition to guiding Chemonics' own programming, we aim to engage with the community of practice in a process of continuous improvement that increases collaborative learning outcomes with global partners and peers. This toolkit will be updated to incorporate new evidence resulting from research-based numeracy interventions being implemented in low-resource countries and contexts. Updates will also incorporate key lessons learned or illustrative good or "best fit" practices from programs implemented in low and middle income countries. The addition of these elements ensures the processes and tools of this toolkit continue to be relevant for the design of effective, low-cost programs that are meaningful for end users.

Contents

Goal

The goal of the numeracy toolkit is to support the design and implementation of comprehensive and responsive numeracy programs and generate impetus for continuous improvement across the global community of practice by sharing this resource to open dialogue, reflection, and collaboration.

In this first iteration, specifically, we aim to equip technicians and program leaders with resources to effectively design and implement quality, inclusive numeracy programs by:

- Increasing understanding of essential concepts of numeracy instruction and mathematics content knowledge
- Providing evidence-based instructional models and effective teaching techniques
- Recommending teaching and learning materials to deliver effective numeracy instruction in low- and medium-income countries
- Increasing awareness of evidence-based best practices for numeracy instruction, including classroom culture that enables the learning environment
- Providing tools, examples, resources, and research that support adaptation to unique country contexts.

Audience

This toolkit is designed for those education technicians seeking to create numeracy programs at the pre-primary and primary levels in lower-resources countries. It is particularly relevant to education technicians from ministries of education and/or implementing partners within the global community of practice. It might also be of interest to civil society organizations and/or nonprofit and private sector enterprises. Written with program leaders and their education technical teams in mind, the toolkit provides research-based guiding principles for quality numeracy instruction and consolidates essential mathematics content knowledge and evidence-based best practices that serve teachers and learners. Additionally, the toolkit contains components to support programming teams in developing processes for systems-strengthening and guidelines for contextualization.

How to Use this Toolkit

The toolkit is organized with chapter-by-chapter resources, tools, and references. Each chapter contains a summary of content and description of relevance to specific users. Individual chapters can serve as a quality assurance resource for team leaders who may not have deep technical expertise in numeracy or mathematics instruction, and to support consensus-building among team leaders and education technicians working on curriculum development or revision. The toolkit can also serve as source of reference material for curriculum developers, and aid master trainers and instructional coaches as a resource for coaching and mentoring instructional leaders and teachers.

Chapter: The (2 + 6) Guiding Principles for Effective Numeracy Instruction

Chapter: Every Learner Can Become Numerate

Chapter: Assessment and Data-driven Decision-making

Chapter: Pathways to Progression; Domains and Higher Order Thinking Skills

Chapter: Instructional Models for Making Mathematics Meaningful

Chapter: Contextualizing Approaches to Comprehensive Numeracy Programs

Annex

User's Note: Toolkit Use

Building inclusive, comprehensive numeracy programs is not a one-size-fits-all initiative. This toolkit is a resource for education technicians to use as a framework for designing new numeracy programs *and* an ongoing resource such that existing numeracy programs can be calibrated with responsive best practices informed by situational analysis of a country context. Segments of this toolkit may be applied and adapted for the context of target populations — community, cultural, and linguistic — including the use of indigenous or traditional numeration systems.

Numeracy and Mathematics

Quick Definitions

The terms numeracy and mathematics are sometimes used interchangeably, but there are key distinctions. The fundamental difference between mathematics and numeracy is that mathematics is the broad **study** of numbers, quantities, geometry, and forms, while numeracy is an individual's knowledge and ability to **apply** mathematical concepts in real life. In essence, numeracy is an individual's literacy in mathematics.

Mathematics refers to the formal study of the measurements, properties, and relationships of numbers, patterns, and shapes, using mathematics concepts, skills, and applications. The field of mathematics has branches, such as arithmetic, algebra, or geometry, which are communicated through standardized symbols and operations, and mastered through learning progressions.

Numeracy refers to an individuals' ability to use, interpret, and communicate mathematical information to reason and solve real world problems. At its most basic form, numeracy involves understanding and applying number sense. More advanced numeracy skills incorporate the use of graphical, spatial, statistical, or algebraic concepts to interpret data and apply it to real world situations.

Humans have demonstrated an innate disposition to develop numeracy[1] (Dehaene and Cohen, 1997). Babies seem to have a preverbal ability to distinguish between quantities (Kucian and von Aster, 2015) and begin to develop numeracy skills at a very early age as they watch and experience the world in their everyday activity and play.[2] As children grow older, their natural numeracy skills deepen as they interact and manipulate objects and concepts, and as they grow confident and learn to manage mathematical facts and principles. While all children are natural engineers, for children who learn the tools and principles of formal mathematics, a new world opens. With these, they can manage more advanced mathematical content knowledge and apply numeracy skills to build a stronger foundation in science, technology, engineering, and better navigate the world equipped with mathematics-based thinking.

Distance Learning and Lessons from COVID-19 School Closures

The year 2020 will be etched indelibly in our minds as the year the world stayed home to try to prevent the spread of COVID-19. Schools around the world shut down for months at a time, leaving students and teachers already struggling with reduced resources at a loss for how to maintain their current knowledge levels and how to extend and grow learning in mathematics and most other subjects. Ministries of education around the world, often with support from donors and other supporters, moved their instructional systems to distance learning to staunch learning loss. In poorer countries and among less resourced families, these transitions were particularly hard, and "blended learning" included the increased use of radio instruction, mobile phones and communication services, such as SMS and WhatsApp, and video and internet when it was available.

However, the ways that children develop numeracy and the need for structured pedagogy, supportive materials, and explicit and systematic learning activities do not change because there is a global

[1] Dehaene S., Cohen L. (1997). Cerebral pathways for calculation: double dissociation between rote verbal and quantitative knowledge of arithmetic. Cortex 33 219–250. 10.1016/S0010-9452(08)70002-9

[2] Kucian K., von Aster M. (2015). Developmental dyscalculia. Eur. J. Pediatr. 174 1–13

pandemic. This toolkit outlines some fundamental aspects of setting up a successful numeracy program for children at the pre-primary and primary levels. While people and platforms for teaching and learning delivery may shift and blend, children still learn numeracy through interacting and manipulating the world around them and building mathematical concepts and language through a series of learning progressions. What we have learned throughout this experience is to stay true to our pedagogical knowledge, and look for new delivery systems and materials for hands-on activities to engage young learners. We also learned the tools we used to fill the gaps the best we could during school shutdown are still available, and with careful attention and intention, we can build successful numeracy programs and series of mathematics curricula to support the ministries and administrators, the teachers and instructional leaders, and the students and families seeking to deliver high-quality, inclusive mathematics programs.

CHAPTER ONE

(2+6) Principles of Effective Numeracy Instruction

This section provides an overview of the (2+6) research-based guiding principles of effective teaching and learning that can drive an effective approach to design and implementation of comprehensive numeracy programs. Although the (2+6) guiding principles are not exhaustive in nature, they represent important elements of programs focused on improving learners' performance in mathematics.

The eight principles are rooted in global research on effective mathematics teaching and learning that supports quality numeracy programs and develops numeracy competencies. The first two principles are foundational to quality teaching and learning, and particularly important for mathematics. The final six principles are specific to numeracy and mathematics instruction. The (2+6) principles are **"what"** evidence-based numeracy programs look like: an aspirational state to improve learners' results that can grow as systems at the classroom, district, and national levels strengthen. Each principle is accompanied by an "Associated Practices" section describing practical ways for **"how"** instructional methods and models, continuums of teacher support, and design of teaching and learning materials integrate to exemplify principles to produce good numeracy programs that improve learning outcomes.

Chapter One will be of interest to:

- **Program design specialists** who design numeracy intervention programs or share knowledge of research-based principles of numeracy teaching and learning, for example, with technical staff at ministries of education or at research institutions.
- **Instructional design specialists** who develop or modify learning instructional materials that support numeracy intervention programs. The "Associated Practices" section outlines types of learning activities that should be woven into these materials.
- **Teacher training specialists** who design continuous professional development programs for evidence-based teaching and learning. The (2+6) principles and associated practices should form the core of mathematics in-service teacher continuous professional development and pre-service programs.
- **Evaluation technicians** responsible for tracking changes in teacher instructional practices. The "Associated Practices" sections describes key practices evaluation specialists can track over time, beginning at baseline, to develop a clear picture of whether teacher practices align with those proven to increase learning outcomes.

Practitioner's Note: Adopt and Adapt What You Need When You Need It

All of the (2+6) guiding principles contribute to effective numeracy instruction and a rich mathematics classroom environment. However, programs may decide to focus more attention on a few priority areas. Chapter 3 provides an overview of data-driven decision-making processes to help identify how to prioritize interventions, and tailor interventions to country-specific needs and resources. Actualizing the (2+6) guiding principles in any numeracy program is never finished, but a continuous process, starting with timing and tuning interventions to the current state and designing the process for improvement toward a more advanced state.

The (2+6) Guiding Principles of Effective Numeracy Instruction

The eight principles are rooted in global research on effective mathematics teaching and learning that supports quality numeracy programs and develops numeracy competencies. As Diagram 1 shows, the (2+6) model positions the first two principles as the cornerstone of all education programs, regardless of subject matter. The first two principles are important when creating an evidence-based, high-quality, and inclusive math-positive learning environment. The remaining six principles are subject matter-specific, and rooted in the research on effective mathematics teaching and learning.

Diagram 1 summarizes the (2+6) principles of effective mathematics teaching and practices associated with each principle.

Diagram 1. The (2+6) Guiding Principles for Effective Numeracy Programs

Each of the (2+6) principles are supported by associated practices shown to improve the quality of mathematics education, and the inclusion of all students across three categories: training, materials, and coaching. While both training and coaching are elements of continuous professional development, their associated practices are listed separately. This toolkit approaches teacher support as a continuum of practice where professional development activities can happen in daily practices at school and include distinct methods to support teachers through periodic learning events or platforms outside the classroom. Contextualizing activities for ongoing improvement of teachers' pedagogical skills that often happen within communities of learning, practicums, or instructional coaching, such as focused observation and feedback cycles or data talks to unpack evidence of student learning and plan the most appropriate response, is a critical component of design and implementation for continuous professional development. A synthesis of this information can be found in Table 5 at the end of this module.

PRINCIPLE 1: Create inclusive classroom environments that foster math-positivity, perseverance, and risk-taking

Students who view math experiences as purposeful and enjoyable will gain confidence and have positive attitudes towards math, and are more likely to be successful in mathematics and to continue their studies.

Research suggests three factors correlate statistically with better mathematical achievement and are directly related to a math-positive learning environment: risk-taking, perseverance, and a collaborative learning environment. Learners are more likely to develop strong mathematics skills in a risk-free learning environment, i.e., when they can share their thinking freely and are not penalized or reprimanded for making errors. To promote this environment, teachers can shift the learner's perspective toward errors or wrong answers as a means of exploring a problem, and as starting point to engage with what learners have misunderstood (Carter, 2008; Kapur, 2011).

For the second factor, perseverance, learners who do not give up when faced with challenging or difficult problems perform better on mathematics assessments (Cuoco et al,.1996, Kilpatrick, Swafford, and Findell, 2001; Hiebert and Grouws, 2007; Hiebert and Wearne, 2003). Learners are more likely to persist or persevere if they have confidence in their mathematics abilities, and when they believe they have the knowledge and skills to solve problems (Kilpatrick, Swafford, and Findell, 2001).

Collaborative classrooms also translate to increased math positivity and inclusivity. Group work provides students with opportunities to explain their thinking and learn from others while actively engaging and practicing mathematics language and logic. Selecting a math curriculum and tools that encourage a growth mindset are effective ways to set up a math-positive learning environment. Empowering teachers through teaching them simple responsive teaching techniques for more productive student interactions, and incorporating regular opportunities for teacher scaffolding, such as hands-on training, coaching, and communities of practice, can help teachers and policy makers build math-positive learning

Gender and ethnicity do not predict math performance

Factors such as gender or ethnicity of a student have shown no measurable impact on a student's ability to succeed in math. While stereotypes often suggest that boys are better at math than girls, studies have shown that any differences in cognitive performance between males and females are most likely the result of social, cultural, and confidence level factors. This is important evidence, because it shows that learners will perform better when they enjoy learning, think what they are learning is useful, and have confidence in their ability to learn. (Fraser and Kahle, 2007) environments. For examples of responsive teaching techniques, see Chapter 2: Every Learner Can Be Numerate.

> [Scaffolding] refers to the steps taken to reduce the degrees of freedom in carrying out some task so that the child can concentrate on the difficult skill she is in the process of acquiring. (Bruner 1978)

Teachers influence how learners perceive their abilities as mathematicians (Cobb and Hodge, 2002; Walshaw, 2004). In mathematics, more than most other subjects, students tend to believe their abilities and intelligence are fixed, and they will either be good at math or not. Evidence suggests that learners develop numeracy skills through hands-on exploration, scaffolding, and practice, and each student does so at a different rate. Teachers can scaffold students to develop a growth mindset where everyone moves forward through an intentional pedagogical process the psychologist Jerome Bruner described as *"the psychology of guided growth*."[3] In classrooms where teachers have a positive attitude towards mathematics, learners also develop more positive attitudes. (Philippou and Christou, 1998). Providing students with positive experiences using math in hands-on ways helps them increase interest, build conceptual understanding, and become more confident in their math abilities.

One of the strongest indicators of academic success is early math skills, followed by early reading skills.[4] Early exposure to practice with numbers, shapes, and quantities contributes to a healthier relationship with numeracy, as well as providing pre-primary educators with the opportunity to connect everyday mathematics with academic math concepts. Common math exercises that favor rote memorization — for example, timed drill and practice — can produce a detrimental level of math anxiety among students (Boaler, 2016). In fact, math taught primarily through the memorization of math facts has been associated with poorer scores on international standardized tests, such as the PISA (Boaler, 2019). An important aspect of a strong numeracy program is creating an inclusive math-positive learning environment that offers a flexible understanding of mathematical concepts and provides the time and tools for students to grapple, explore, and practice before committing math facts to memory.

ASSOCIATED PRACTICE

TRAINING: Raise teacher awareness of the importance of fostering a math-positive environment and growth mindset among learners. Demonstrate the correlation between learners' attitudes and self-image about mathematics and their mathematics performance. Demonstrate how to set up a collaborative classroom and activities that explicitly and systematically build mathematics skills through exploration and hands-on learning.

Mathematics programs should introduce teachers to research on the effects of teacher (and learner) attitudes on learners' willingness to engage and persist in mathematics learning, and their perceptions of their mathematics abilities. Classroom practices that foster risk-taking, perseverance, and collaboration include:

- Displaying a growth mindset about mathematics for all learners
- Communicating that all learners can be successful mathematicians if they work at it

[3] McLeod, S. A. (2019, July 11). Bruner - learning theory in education. Simply Psychology. https://www.simplypsychology.org/bruner.html

[4] Duncan, G. J., Dowsett, C. J., Claessens, A., Magnuson, K., Huston, A. C., Klebanov, P., Pagani, L. S., Feinstein, L., Engel, M., Brooks-Gunn, J., Sexton, H., Duckworth, K., & Japel, C. (2007). School readiness and later achievement. Developmental Psychology, 43(6), 1428–1446. https://doi.org/10.1037/0012-1649.43.6.1428.

- Communicating that it is more important to make progress than to quickly achieve milestones
- Insisting learners can and will solve problems if they stick with them, and explore different ways to understand problems
- Not penalizing or reprimanding learners for making mistakes; instead, communicating that mistakes are tools to help us identify what we have misunderstood
- Encouraging all learners to engage through collaborative group work and exercises in creative approaches to problem-solving in math activities and discussions
- Creating activities that address the needs of learning and arouse their interest and curiosity.

SUPPORT/COACHING: Provide structured opportunities to scaffold and support teachers as they test out these practices in the classroom on a daily basis. Demonstrate how positive feedback, collaborative classrooms, and a growth mindset can increase perseverance and risk-taking.

Simple tools for coaches that collect classroom data can encourage math success, perseverance, and risk-taking. Coaches and instructional leaders should work with teachers to identify math stereotypes, eliminate practices that reinforce them, and introduce accommodations and activities to enable students that are moving more slowly to continue practicing and learning concepts. Coaches should analyze the data on teacher-initiated interactions, collaborative activities, and methods to foster math-positivity with teachers, with a view to identifying areas of strength as well as areas that need improvement. By tracking these data-based results over time, coaches and teachers can determine whether classroom learning environments are evolving as expected.

PRINCIPLE 2: Assess regularly to identify and fill learning gaps and teach responsively

Teachers are more effective when they regularly assess progress and customize instruction to address learning gaps and deepen understanding.

Practitioner's Note: Formative assessment is one of the crosscutting practices of any education intervention, particularly for effective numeracy programs. Formative assessment is important to instructional designers when planning how to integrate systematic progress monitoring and checks for understanding into teaching and learning materials. It is critical that teachers are supported to regularly look at evidence of student learning to understand student progress and respond appropriately so that no learner is left behind. More detailed information on practical ways to integrate formative assessment and associated practices into everyday teaching and learning can be found in Chapter 3.

Designing mathematics curricula and teacher training based on research helps lay the foundation for classroom success. Equally important is for educators to capture regular real-time data to determine if students are learning, and how they should adapt instruction. Gathering information can be as simple as informal assessments that balance careful listening with thoughtful questioning. Teachers do this when they intentionally watch or monitor learners' thinking during class discussions or practice time. Teachers can use this information to decide whether to step in (or out) of the discussion, press for understanding, or address misunderstandings or confusion (Lobato, Clarke, and Ellis, 2005). Careful listening and purposeful questioning provide a powerful way to understand what learners already know and what they need to learn next (Walshaw and Anthony, 2008; Steinberg, Empson, and Carpenter, 2004). In turn, teachers can then decide whether to move on or review key understandings (Franke, Fennema, and Carpenter, 1997, as cited in Franke et al., 2009). Even in crowded and mixed-level classes, a teacher-led formative assessment provides an important snapshot of information and strategies of the learners' progress, and is critical to help teachers reorganize their instruction to teach based on evidence.

Wiliam (2010) notes three processes in formative assessment: (1) Identify where learners are; (2) Identify goals for learners; (3) Identify paths to reach goals.

NAME: Mugisha, Prossy

PLACE VALUE	NOT THERE YET	ON TARGET	ABOVE AND BEYOND	COMMENTS
Skip counts by tens		✓		
Skip counts by hundreds		✓		Took out hundreds pieces while skip counting
Understands 100 equals ten tens	✓			
Reads and writes numbers to 1000	✓			
Compares three-digit numbers		✓		Showing greater reasonableness
MATHEMATICAL PRACTICES				
Makes sense of problems and perseveres		✓		Stated problem in own words
Models with mathematics	✓			Reluctant to use abstract models
Uses appropriate tools		✓		

Formative assessment is a powerful crosscutting educational principle, and particularly so when linked to mathematical learning progressions (see Principle 3). Formative assessment systems work best when there is a combination of regular data-gathering using tests of achievement and simple teacher-led practices and tools, support for a math-positive learning environment where mistakes are not stigmatized but are instead seen as opportunities for learning, and available follow-up suggestions and advice for the teachers on how to make adjustments. As well, assessing milestones along a learning progression is crucial, because learners learn at different rates, and need a deep understanding of certain concepts before they can make progress (Clements, Sarama, and DiBlase 2003).

ASSOCIATED PRACTICES

TRAINING: Train teachers on simple formative assessment practices, including tools to monitor progress through observation, anecdotal notes, checklists, questions, interviews, and tasks with respect to milestones, while differentiating instruction based on the results.

Teacher support for in-service and pre-service programs should address strategies for a three-pronged approach: regular assessments to monitor progress, math-positive environments where wrong answers are not penalized, and inputs for teachers on how to follow up assessment results with differentiated strategies. This type of continuous assessment is several degrees more formal, but closely related to the daily checks for understanding that happen during everyday classroom instruction that can create a math-positive environment. Ideally, data from both types of learner feedback inform discussions between teachers and instructional leaders on how to best respond to learners' needs. Training programs should highlight the integral role of assessment and follow-up classroom support activities — such as making time for explicit reteaching or identifying alternative means of expression appropriate for the next lesson — in lesson plans, classroom monitoring, and community of practice sessions.

MATERIALS: Provide teachers with simple tools to measure, at key points in the school year, learner progress toward key milestones along learning progressions, as well as simple, follow-up activities differentiated according to results.

Structured or model lessons can systematically allocate five to 10 minutes of a lesson to independent practice. During these moments of a lesson, learners will independently solve problems or tasks related to the covered concepts and record their strategies and answers. Materials can detail key elements that teachers should observe in learners' work, including common errors and how to best correct them (see textbox), as well as key questions to ask to consolidate or reinforce learning.

Quick Assessment: Looking for common number value errors

- Writing 37 as 307 or reading 307 as 37 (in English).
- Writing sixteen as 61
- Thinking that .67 is bigger than .8 because it has more digits
- Thinking that ¼ is bigger than ½ because 4 is bigger than 2

Instructional materials should identify intermediary "milestones" or learning outcomes that learners must be able to demonstrate at key points. These milestones should be linked to the mathematical learning progressions in the curricula, and include simple, easy-to-administer and interpret, low- or no-cost formative assessment tools to measure them. This will form the basis for follow-up differentiated instruction sessions for learners who have not met the milestones with activities to address misunderstandings or learning gaps. This time can also provide opportunities for more practice or enrichment for learners who have met milestones. The proposed pace of learning in the instructional materials ideally build in time for both the formative assessment and the differentiated instruction, ensuring that all learners have the needed understanding to move to the next milestone in the progression.

SUPPORT/COACHING: Support implementation of simple formative assessment practices during lessons and use milestone assessment tools to monitor progress and institute follow-up differentiated learning opportunities

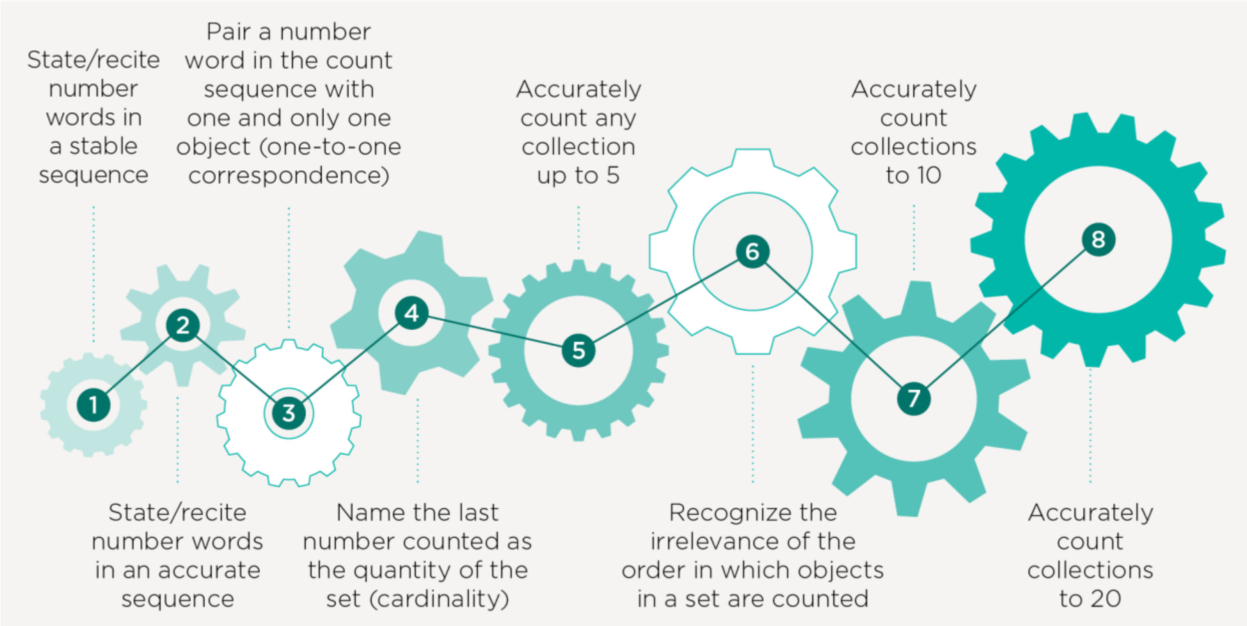
Work with instructional leaders and coaches to design and integrate simple instruments, such as 10-minute data talk protocols, to monitor teacher interactions with respect to their use of formative assessment tools. The focusses will be helping teachers to institute targeted differentiated instruction, based on assessment results, and maintaining a learning environment where incorrect answers are opportunities for learning and discussion.

PRINCIPLE 3: Respect Learning Progressions

Learners learn better when instruction follows evidence-based cognitive learning progressions and the pace of learning allows for the development of deep understanding and practice.

Using textbooks and teacher training materials that carefully scope, sequence, and structure learning is particularly important in mathematics education. Cognitive research across cultures indicates that children's mathematical thinking goes through identifiable stages of conceptual development (Clements, 1999). Sometimes called learning progressions — or learning trajectories — these are the cognitive pathways children follow as their thinking becomes more sophisticated and complex. The steps in a learning progression vary according to the mathematics domain, or the concepts within a domain. An example of a learning progression for learning and its milestones, taken from Instructional strategies for Mathematics in the Early Grades (Sitabkhan et al., 2019), is shown in Diagram 2 below.[5]

Diagram 2. Example of a Developmental Progress for Counting to 20

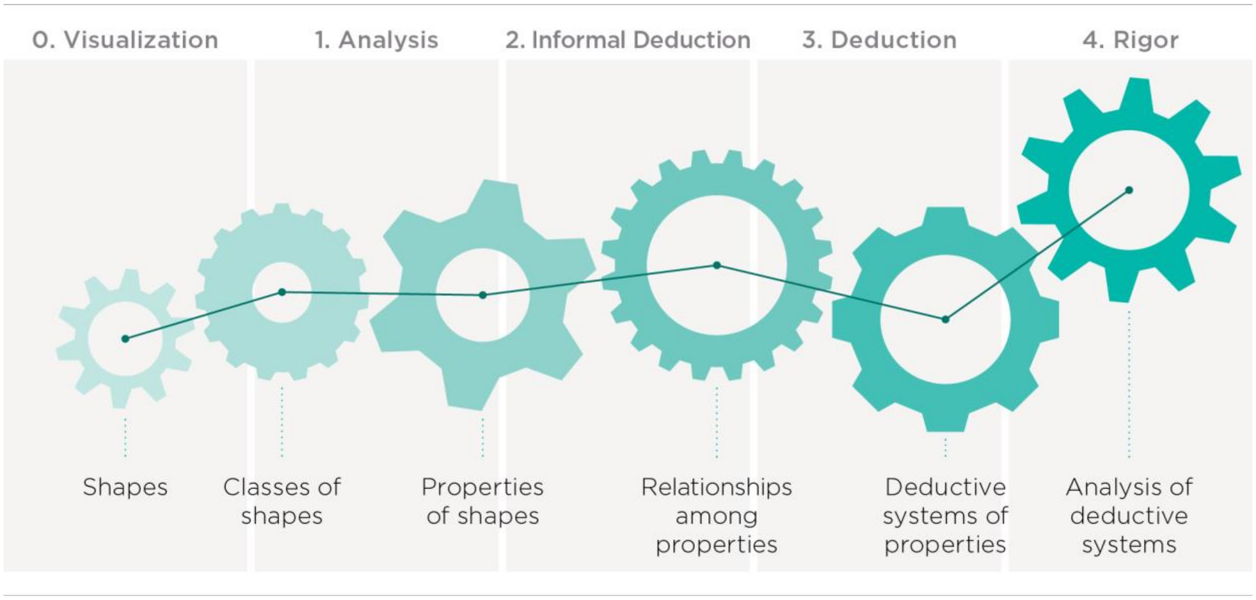


It is important to note that learners do not move automatically from one level to the next simply as they age. Learners move from one level to the next if they engage in carefully selected bridging activities that move them gradually and systematically from their current level of understanding to the next level. Figure 3 below, summarizes the five Van Hiele levels of geometric thought, i.e., the mental processes involved in mastering basic concepts in geometry. The lowest level, visualization (designated as level 0 in Van Hiele's progression), refers to learners' ability to identify shapes by their form. At this level, for example, a learner may recognize that a shape is a rectangle because it looks or feels like a shape called a rectangle. At the next level, Analysis, learners can identify the properties of the shape. Learners develop these concepts at each level as they are introduced to new shapes and build the foundational mathematical thinking for geometry.

[5] For a more complete discussion of developmental progressions in early primary, see Clements, D. & Sarama, J. (2010). Learning Trajectories in Early Mathematics – Sequences of Acquisition and Teaching. Encyclopedia of Early Childhood Development or Daro, Phil; Mosher, Frederic A.; and Corcoran, Thomas B. (2011). Learning Trajectories in Mathematics: A Foundation for Standards, Curriculum, Assessment, and Instruction. CPRE Research Reports. Retrieved from http://repository.upenn.edu/cpre_researchreports/60. The appendix of the latter document gives an overview of key learning progressions for mathematics.

While progressions within some domains are linear, the Van Hiele learning progressions for geometric thought are not linear, and a learner will go back and revisit different stages of thought when new shapes are introduced as they become more familiar with geometric concepts. Students also cannot "skip" a level. This process of revisiting previous stages and fine-tuning concepts is a good example of how students become expert learners. While the levels of geometric thought may sound highly technical, the underlying concept of learning progressions is important to mathematics instruction.

Diagram 3. The Five Van Hiele Levels of Geometric Thought



When mathematics curricula and instructional materials developers sequence learning activities according to research-based progressions, learners can gradually move from simpler concepts to more complex ones and gradually and incrementally deepen their thinking (Clements and Sarama, 2014; Flowers, et al., 2007). Because learning improves when the pace of instruction allows learners to develop a deeper understanding, building in the time for this to happen generally requires reducing the number of topics covered in a given year (i.e., the breadth of topics covered). Ministry officials, curriculum designers, trainers, coaches, as well as teachers should understand the importance of a sequenced and structured teaching and learning package that is aligned with how children's development will progress over time.

Many recently developed curricula in high-resource countries, for example, devote considerably more time in second grade to place value concepts before tackling the addition, subtraction, multiplication, and division of multi-digit numbers and decimals in subsequent grade levels. They will also sequence geometry instruction to gradually move learners through the different levels of geometric thought and delay the introduction of fractions and decimals, and operations on these numbers, until students establish mastery. These approaches allow learners develop a deep, nuanced understanding of these concepts.

Practitioner's note: Consider context when forming and messaging expectations

Although all learners progress, they do so at different rates. By carefully listening to a student's thinking, teachers who are familiar with developmental progressions for key concepts are able to determine which learners are ready to move on and those who are not. This type of formative assessment helps teachers differentiate instruction, ensuring struggling learners receive more time and support to master key concepts. Supporting teachers coping with large class sizes in this process means designing teaching and learning materials packages that are explicit and systematic ***and*** encourage and enable teachers to make decisions about practical modes of differentiation that do not overly rely on homogeneous grouping.

ASSOCIATED PRACTICES

TRAINING: Train teachers on cognitive learning progressions in key concepts for targeted grade levels, and how to use instructional materials that align with those learning progressions.

Introducing Learning Progressions to Teachers: An Example

A simple but effective method for introducing the importance of learning progressions is to provide groups of teachers with a list of key steps in a learning progression for a key concept and have them work together to arrange them in a sequence of increasing difficulty. These types of activities require teachers to think deeply about a given concept and the stages learners go through to understand it. Having groups compare their sequences and come to a consensus on what they perceive to be the "optimal" sequence can deepen their understanding even further.

Teachers who are aware of developmental progressions understand how to interpret teachers' guides, weekly schemes, and lesson plans that sequence instruction to optimize learning and how to respond to assessment data. In-service and pre-service mathematics programs should introduce teachers to learning progressions for key math concepts within targeted grade levels (see text box) and provide support through opportunities to practice slowing down or speeding up along a progression in response to learners' feedback. Well-supported teachers see learners through two lenses — growth according to progress along a progression and performance according to grade level competencies — and use their understanding of both to understand and respond to learners' needs.

MATERIALS: Develop carefully sequenced math curricula and/or instructional materials that reflect research-based learning progressions.

A useful starting point for discussions around curriculum or instructional materials is research on developmental learning progressions, and their role in improving mathematics learning outcomes. Chemonics is committed to supporting the development of curricula, scope, and sequences[6] and related mathematics instructional materials that align with global research on cognitive learning progressions and the appropriate paces of learning.

Once research-based curricula and associated scopes and sequences are established, they can be translated into easy-to-use teacher's guides (and/or associated learner workbooks or textbooks) that:

[6] The Global Proficiency Framework for Mathematics to support student learning and reporting on Sustainable Development Goal 4.1.1, provides a general overview of progressions for each domain of mathematics, from grades 1 to 9. This document can be a useful starting point for identifying grade-appropriate learning outcomes for each domain of mathematics. However, the document does not provide the level of detail required to develop a scope and sequence for research-based instructional materials.

- Indicate the milestones, including the intermediary ones, that learners must meet with respect to each learning progressions
- Lay out a clear, explicit, evidence-based scope and sequence based on learning progressions
- Provide teachers with simple, but carefully structured activities to meet each of the intermediary learning outcomes in the progression
- Pace learning appropriately and encourage teachers to adjust pace according to learners' needs
- Provide teachers with simple formative assessment tools and techniques to measure progress against milestones and appropriate remediation/differentiation activities to implement, based on the results.

SUPPORT/COACHING: Support teachers to use evidence-based instructional materials in the classroom based on learning progressions and associated instructional practices

Teachers need ongoing school-based support and coaching, particularly when new instructional materials (e.g., teachers' guides, learner workbooks/textbooks, supplemental materials, manipulatives) are distributed. Instructional leaders and coaches can conduct, co-plan or co-teach model lessons to reinforce the use of activities in structured lesson plans. Teachers can participate in community of learning activities to reflect on components of teachers' guides and evidence of student learning to modify delivery of lesson plans or activities within a weekly scheme to meet learners where they are at and maximize engagement and instructional time. Coaches can also collect data on the extent to which teachers are able to follow the structured lessons and guidelines or use the formative assessment activities as intended.

PRINCIPLE 4: Connect formal and informal mathematics

Students learn better when teachers explicitly connect formal school mathematics to the informal mathematics learners use and explore outside of school.[7]

Informal mathematics Informal mathematics skills are those learned before or outside of school, often in everyday situations, including play, and often without formal mathematical symbols or equations (Seo and Ginsburg, 2003). In fact, babies are born with a basic understanding of quantities (Starkey and Cooper 1980; Strauss and Curtis 1981; Berger, Tzur, and Posner, 2006) and can distinguish two objects from three, and sometimes even three from four. By 30 months, most infants can do some basic counting, without having been explicitly taught (Wynn, 1990), and are able to answer simple "how many" questions. This learning progression generally develops without any explicit or formal teaching of mathematics (Sousa, 2008).

When learners are confronted with mathematics-related problems outside the classroom, they draw solutions from a variety of creative strategies and resources. Young children develop informal mathematics skills through participation in everyday math-related activities, for example, when they have to count objects, sell or purchase items in the market, or help construct an animal enclosure. They also evaluate the correctness of their solutions based on whether they have successfully solved the problem (Hoyles et al., 2001; Martin and Gourley-Delaney, 2014). These important mathematics processes and

[7] For a review of the literature on the importance of connecting formal and informal mathematics, see Patterson, Rubin and Wright (2017), Mathematics in informal learning environments: A summary of the literature, https://www.informalscience.org/sites/default/files/InformalMathLitSummary_Updated_MinM_03-06-17.pdf accessed December 2020.

skills, developed naturally as learners engage in informal mathematics, are critical to the formal study of mathematics.

Formal mathematics Formal mathematics uses standardized written symbols (i.e., numerals) and models (i.e., objects and diagrams like the number line) to represent abstract mathematical ideas. Use of standardized symbols or models is a critical distinction between informal and formal mathematics and enables learners to generalize what they have learned and apply those generalizations to new problem situations.[8] For example, learners who have developed a formal understanding of the algorithm for subtracting two-digit numbers will use that understanding to subtract three-digit numbers (Sitabkhan et al., 2019). (See Instructional Models for Making Mathematics Meaningful).

Connecting the two Connecting informal mathematics learners do outside of class to the formal mathematics done in the classroom is critical to helping learners make sense of formal mathematics (Martin and Gourley-Delaney, 2014). This is particularly important in the early years, as foundational concepts such as number sense, simple operations, and fractions are concepts learners encounter frequently in their everyday activities. They learn, for example, how to solve “fair-share” problems, such as sharing a number of sweets equally among their siblings, an informal mathematics situation that links to the formal mathematics concept of division. Making explicit connections between formal mathematics and informal mathematics learners use in everyday life communicates to them that the knowledge they bring to the classroom is valuable. This reinforces the understanding that mathematics is important and relevant to everyday life.

Users’ note: The reader will find more information about the development of informal mathematics in young children and their importance in supporting children’s understanding of formal mathematics in **Instructional Models for Making Mathematics Meaningful.**

ASSOCIATED PRACTICES

TRAINING: Raise teacher awareness of the importance of connecting informal and formal mathematics, and provide them with instructional time and practice using tools and strategies. Supportive programs will introduce teachers to simple instructional strategies that help make those connections more explicit, including:[9]

1. **Bridging**, or teacher-led discussions where teachers explicitly connect a problem in class to an out-of-school problem. When teachers connect problems in class to an everyday act like receiving change when at the market, they can give meaning to an abstract problem.

[8] One of the important distinctions between formal and informal mathematics is the ability to generalize. In informal mathematics, learners develop strategies to solve the problem at hand. However, because the strategies are situation-specific, learners are not able to generalize their solutions to a wide range of similar situations. Formal mathematics, on the other hand, uses standardized symbols and models so that learners can generalize their understanding to a wide range of situations.

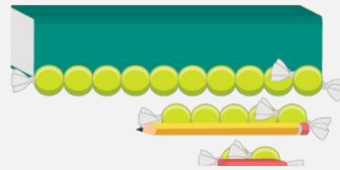
[9] The two strategies referenced above were proposed by Sitabkhan et al., in Instructional Strategies for Mathematics in the Early Grades (2019).

2. **Introducing word problems that connect to mathematics learners engage in outside of school.** which relate to an activity or an event occurring in the classroom, the school, or the community. For example, learners having to share textbooks in the class, or organizing transportation for an upcoming visit to a wedding.[10]

3. **Linking non-conventional units of measurement with formal mathematical units,** providing students with an explicit understanding of the relationship between personal computational strategies and formal approaches. (see text box).

Example of measurement

This example demonstrates one non-conventional way of discovering units of measurement. Learners use one item — in this case, a sweet — to determine the length of an eraser, a pencil and a book.



MATERIALS: Ensure new mathematics instructional materials provide space to ground problems in real life situations.

Informal applications serve as a springboard for the introduction of key concepts when using instructional models and materials. Model lessons in teachers' guides, for example, may begin with an exploration or discussion of everyday math-related problems related to the target concept. Textbooks or workbooks may include an example or picture of a young person engaging in a related informal learning activity before introducing the formal concept. Education technicians can identify how learners at each targeted grade level engage with key mathematics concepts in their everyday lives. This also offers a pathway for inclusion, as children who may have learning gaps according to the milestones for their grade level, can likely all relate to social, cultural or geographical examples that can connect to the mathematical concept being taught.

SUPPORT/COACHING: Provide teachers with in-class support on how to link informal and formal mathematics in the classroom.

Numeracy programs can include in-school support and follow-up activities for teachers that link informal and formal mathematics through bridging and word problems. This can be a particularly rich discussion for communities of learning, where informal connections can be made meaningful in a specific context. It can also be a point of discussion for coaches and teachers prior to observation and in post-observation reflections.

PRINCIPLE 5: Promote learning through problem solving

Learners learn better when they have opportunities to learn through problem-solving and are taught evidence-based strategies to solve problems.

[10] Sitabkhan et al. (2019) cautions that in order to make the informal-formal connection, the word problems must represent actual situations children encounter, as opposed to applications of a concept. They give the example of a problem like: "There are eight children on a bus. Five are boys, and the rest are girls. How many girls are on the bus?" While the problem may appear to be connected to the mathematics in learners' lives, unless learners regularly take the bus and calculate how many girls are it, it is not connected to the mathematics they do outside of school. Rather, the problem falls into the category of application problems, where learners apply a concept learned in class.

Learners are confronted with different types of problems to solve in their daily lives, but in mathematics instruction, not all problem-solving is the same.

What makes a problem routine or not routine?

Whether or not a problem is routine or non-routine depends upon the knowledge learners bring to the table.

For learners who have not been formally introduced to the way to set up an addition equation, a problem such as “Mary has four cookies. Abdullah has two cookies. How many do they have together?” will be a non-routine problem. For learners who have been formally introduced to how to set up addition equations, such a problem will be a routine, application-type problem.

Routine problems are problems that have an easily identified solution path and a single answer. Most routine problems are *application exercises*. They require learners to apply a previously learned procedure, such as adding two numbers to find a total, or applying a concept in a known manner. For that reason, they are generally not considered to be “problems" in the true sense of the word.

Common underlying structures of addition and subtraction type problems in early primary

Change type problems where the result is unknown, when a quantity is increased or decreased by a quantity. For example, “Abdul has five candies. His sister gives him three more. How many does he have now?”

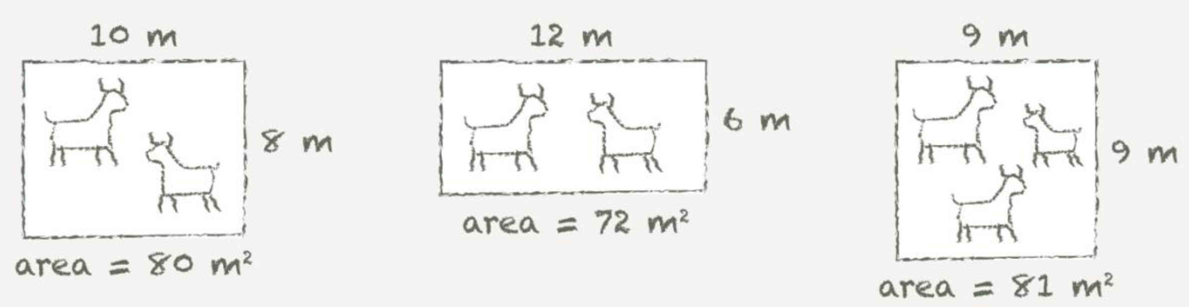
Change type problems where the initial quantity is unknown. For example, “Abdul had some candies. His sister gave him three more. He now has eight. How many candies did he have in the beginning?” Or “Abdul had some candies. He gave three to his sister. He now has eight. How many candies did he have in the beginning?”

Change type problems where the quantity added or removed is unknown. For example, “Abdul had eight candies. He gave some to his sister. He now has five. How many candies did he give to his sister?” Or “Abdul had eight candies. He sister gave him some candies. He now has 10. How many candies did his sister give him?”

Comparison type problems. For example, “There are five boys on the soccer field and eight girls. How many more girls than boys are there on the soccer field?”

Non-routine problems are problems without an easily identifiable procedure, and may have more than one way to solve the problem. For example, if a student is asked to determine the optimal dimensions of a four-sided fenced area to hold animals and is given a set length of fencing, say 36 meters, they may discover that four-sided shapes with the same perimeter (in this case, 36 meters) can have very different surface areas. They may also discover that the surface area increases as the difference between the length and width of the four-sided shape decreases. These discoveries naturally lead to the understanding that, for a given perimeter, a square shape will have the greatest area, and why (see text box).

Relationship between side lengths of 36 meter enclosures and surface area enclosed



Learner-led problem-solving Learners acquire their understanding of mathematics and develop strong skills when they solve problems (Hiebert et al.,1977), as opposed to being "taught" mathematics. This approach is very different than traditional top-down routine problem- solving, where teachers present a problem, explain to learners how to solve it, and then ask them to use the strategy presented to solve similar types of problems.

Link to other (2+6) principles

Learner-led problem-solving includes having learners share their thinking (or solution paths) and answers with others (Principle 6), preferably using models (manipulatives or diagrams) (Principle 7) and listen to and respond to ideas or models presented by others by asking questions or identifying flaws in the thinking (Principle 2).

Teachers who adopt a learner-led problem-solving approach begin their lessons by presenting learners with an interesting problem related to everyday life experiences, and ask learners to work with a partner or small group to try to solve it.

Link to mathematical habits of mind When learners engage in learner-led problem-solving, they develop mathematical ways of thinking and reasoning, often referred to as *mathematical habits of mind.* Mathematical habits of mind are nurtured when teachers encourage learners to persevere when they do not get the answer immediately (see Principle 1), and when teachers provide positive reinforcement to learners for identifying innovative or different ways of solving things. This creates an opportunity for a teacher to responsively integrate an unscripted and explicit moment that can cue all learners in the class to a desired mathematical behavior. When facilitating learner-led problem-solving activities within a lesson, a teacher might briefly say something to redirect or reinforce, by saying something like, "I see that the red team is adjusting their drawing to show the answer, after they used their sticks to check their answer and found it was wrong. This is how good mathematicians solve problems. We try different ways until we get it right."

ASSOCIATED PRACTICES

TRAINING: Introduce teachers to the power of non-routine problems, the importance of a learner-led problem-solving approach, and key activities that can guide the process.

When teachers have first-hand experience of the power of learner-led problem-solving approaches in math classes, they are more likely to encourage learners to learn by doing through solving rich problems (non-routine problems) rather known formulas (routine problems). Whether it happens through self-study with distance support, in a community of learning, or at a practicum, we recommend that teacher training incorporate hands-on activities to practice problem-solving where teachers work to solve rich, meaningful problems linked to a key concept, that can be solved in multiple ways. As teachers work with a partner or a small group to solve the problem, facilitators can encourage them to persevere or try another strategy if they run into difficulty.

During the follow-up stage of a training exercise, facilitators can model a guided discussion to have teachers compare their solutions, solution paths, and new understanding. Facilitators will ask carefully phrased questions to help learners clarify and expand on their thinking and guide them to identify the specific math concepts explored in the problem and how they connect with other problems or situations encountered. As a final stage in the learning process, facilitators will provide teachers with new problems that draw on the new knowledge gained.

After experiencing the three-step learner-led problem-solving instructional model, facilitators will debrief the learning experience with teachers to identify *what* they learned and *how* they learned, including what enhanced or inhibited their progress. Such discussions raise awareness of the power of learner-led problem-solving and the roles of teachers and learners in the three-step instructional process.

The approach described above has the added benefit of strengthening teachers' own conceptual understanding of critical mathematics concepts in a low barrier environment. This is often necessary in contexts where teachers have nascent conceptual understanding or lack confidence in their mathematical abilities. Teachers are unlikely to introduce learner-led problem-solving if they think they will be unable to understand pupils' solutions or their different solution paths.

MATERIALS: Ensure that new instructional materials include models for learner-led problem-solving.

Teachers are more apt to adopt learner-led problem-solving if they have access to rich problems that are aligned with important mathematics concepts in the curriculum and the steps in an associated learning progression. In ***Instructional Models for Making Mathematics Meaningful***, information on three formats of mathematics lesson plans are presented, along with weekly schemes that demonstrate explicit and systematic delivery of content according to a scope and sequence. Here we emphasize that the learner-led problem-solving model is a particularly important for math class, which flips ***I do, we do, you*** do approach and considers constraints large class sizes.

We recommend that education technicians develop exemplar problems aligned with important mathematics concepts that are heavily weighted within the competency framework at each grade level and that they modify or supplement existing textbook problems to make them richer and more engaging mathematically We also recommend they develop model lessons that provide teachers the scaffolded support required to implement these important practices in their classroom. Where resources permit, we encourage education technicians and the governments they support to develop detailed teachers' guides with model lessons that systematically include learner-led problem-solving *and* other instructional models, and build the understanding that there are opportunities for teachers to be explicit and responsive in every model.

SUPPORT/COACHING: Support teachers to implement learner-led problem-solving model lessons.

Where learner-led problem-solving is a key component of numeracy programs, education technicians are encouraged to develop simple data collection instruments to understand the extent learner-led problem-solving is evident in teacher practices, and whether it is being implemented effectively. When teachers have been provided with teachers' guides and model lesson plans, data collection may focus on measuring how teachers are able to follow these plans, and analyzing reasons for any deviations, as well as simple protocols for analyzing evidence of student learning.

PRINCIPLE 6: Encourage learner "talk" and the explanation and justification of mathematical reasoning

Asking learners to explain why they chose a strategy and how it worked, or to analyze other strategies to solve a problem, helps promote deeper mathematical understanding.

Young children learn mathematics by engaging in problem-solving, by talking about what they are doing and thinking as they solve problems, and by reflecting on their actions and the actions of others (Ministry of Education, Ontario, 2003, p. 10). When learners explain, justify, and debate their strategies for solving math problems, or analyze the strategies used by others, they deepen their mathematical understanding (Trocki, et al., 2014; National Research Council 2001; Hiebert and Wearne, 2003).

Importance of explanation and justification

"Describing, explaining, and justifying one's thinking helps students internalize principles, construct specific inference rules for solving problems, and become aware of misunderstandings and lack of understanding." (Chi, 2000, 2009)

Research supports the need for mathematics classrooms, beginning in pre-primary, to be dialogue and conversation-rich (Cyr 2011; Franke et al. 2009; Trocki, Taylor, et al., 2014; National Research Council, 2001). Asking learners to explain why they chose a strategy and how it worked or to analyze other strategies promotes deeper mathematical understanding (ibid.; Hiebert and Wearne, 1993). These questions scaffold learners' engagement with the task, shape the nature of the classroom environment, and create opportunities for learning high-level mathematics (Boaler and Brodie, 2004; Kazemi and Stipek, 2001; Stein, Remillard, and Smith, 2007, as cited in Franke et al., 2009).

Teachers who ask open-ended questions or encourage students to ask questions of each other help learners think critically about their mathematical approach (Cyr, 2011). Teachers who recognize the value of having learners "talk" tend to use learner-centered questioning techniques. Teachers who value learner "talk" also understand that it is a particularly effective strategy for identifying what learners have misunderstood and helping them correct their misunderstandings. They spend less time explaining things and more time giving learners rich problems to solve,[11] having them solve them using materials or diagrams (see Principle 8), and then asking questions that require learners to explain what they learned: the strategies used, the relations discovered, and the answers found.

This is particularly true when teachers ask learners to explain incorrect or incomplete solutions. Doing so communicates that making mistakes is part of the learning process. These discussions should be delivered with techniques that normalize error and create emotional constancy (Principle 1) for correct and incorrect solutions (See ***Every Learner Can Be Numerate***.)

ASSOCIATED PRACTICES

TRAINING: Train teachers on the value of learner "talk" and how to use simple, generic questions to encourage such talk in the classroom.

Numeracy programs should include activities to introduce teachers to the research on the value of learner "talk" and to model lessons that demonstrate how simple, generic questions (see text box) can encourage learner-talk in the class. Follow-up with micro-teaching sessions during trainings and practice among peers.

[11] Gadanis, Hughes et al. 2009, cited in Cirillo 2013, stress that the relationship between good tasks and good discussions is simple: If we want learners to have interesting discussions, we need to give them something interesting to discuss. Simply put, it is easier to have learners engage in meaningful discussions of their thinking and learning if teachers give them tasks or problems that can be solved using multiple strategies, that address core mathematical ideas, and that are of interest to learners (Franke, Kazemi, and Battey 2007).

MATERIALS: Provide teachers with instructional materials that model how to facilitate learner-led talk in the classroom.

Evidence suggests that learners are more apt to benefit from "talking" about their thinking when: 1) they are given rich, interesting problems that all learners can solve, 2) all voices are encouraged and heard in the classroom, and 3) all learners have opportunities to contribute to classroom discussions (Hiebert et al., 1997). Education practitioners should design teacher support materials that include model lessons and outline a series of key guiding questions that teacher can ask at different points (see text box).

Questions that encourage learners to talk about their thinking and reflect on the "talk" of others

- How did you get that answer?
- Can you say more about….?
- Why did you choose to use…?
- What other strategy could be used?
- Do you agree with (Pupil A's) answer? Why or why not?
- Can someone say, in their own words, what (Pupil A) just said?
- Let's hear what other pupils are thinking about that…

COACHING: Support teachers to increase learner-led talk in the classroom by asking questions that promote explanation and justification.

Model lessons given in this toolkit, when supplemented by in-class visits from coaches, can support teachers to gradually shift their classroom interaction patterns from teacher-led to learner-led. As part of the coaching process, coaches can collect data to provide teachers with objective feedback and facilitate reflection on how their classroom interaction patterns are shifting. This may include finding the percentage of learner versus teacher talk in the lesson, the number and types of questions asked to encourage learners "talk," or the effective teaching techniques to build trust and increase engagement.

PRINCIPLE 7: Use manipulatives, tools, and models intentionally

Using manipulatives and symbols to represent mathematical ideas is correlated with increased learning outcomes.

There is a solid research base, stretching back to the mid-1950s, demonstrating the importance of using objects or diagrams to explore or express abstract mathematic ideas.

Bruner's research in the 1960s demonstrated that young learners learn mathematics by doing, i.e., by using concrete objects to construct physical models of abstract mathematic ideas (Bruner, 1966). Meta-analyses of more than 100 studies conducted by Suydam and Higgins (1977), Parham (1983), and Sowell (1989) found that long-term use of objects — often referred to as manipulatives — to represent mathematical ideas correlates with increased learning outcomes. A more recent review (2001) by the National Research Council (see *Adding it Up: Helping Children Learn Mathematics*) concluded that "manipulatives can provide valuable support for learning when teachers interact over time with (learners) to help them build links between the object, the symbol, and the mathematical idea both represent" (p. 354). A subsequent metanalysis carried out by Gersten et al., (2009)[12] identified visual models and manipulatives as a research-based practice for improving mathematics learning outcomes.

[12] See report "Assisting students struggling with mathematics: Response to Intervention for elementary and middle schools" (2009) available at https://ies.ed.gov/ncee/wwc/Docs/PracticeGuide/rti_math_pg_042109.pdf#page=32

Hands-on, manipulatives-based activities can help learners give meaning to new mathematical knowledge or concepts (Stein and Bovalino, 2001). Manipulatives and diagrams provide learners with a shared, simple way of talking about concepts or communicating their understanding. They fill in gaps in verbal explanations, allowing young learners to "show" their thinking. For example, in pre-primary and early primary, counters help learners understand and communicate what we mean by "three" or "two plus three." In middle-to-late primary, a simple ball — even a fruit like an orange — can help learners understand the difference between the circumference and the diameter, or the shape produced when a sphere is sliced open.

Examples of math manipulatives

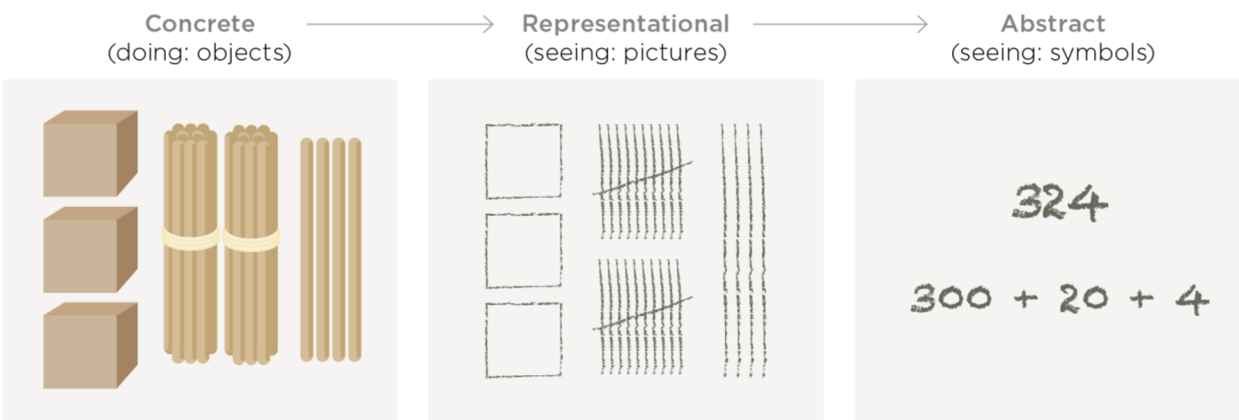
Manipulatives are physical objects designed to represent explicitly and concretely abstract mathematical concepts (Moyer, 2001).

Finally, hands-on, manipulatives-based learning activities can increase learners' interest in and enjoyment of mathematics (Sutton and Krueger, 2002), which correlates with increased mathematical ability (ibid.). As learners' thinking progresses, they tend to move from representing mathematical concepts with concrete objects they can hold, move, and put together to representing these same concepts with diagrams and pictures (e.g., number lines or hundreds charts), and finally, with symbols.

The Concrete-Representational-Abstract (CRA) instructional progression. Bruner's research[13] (Bruner, 1966) led to the development of a three-step evidence-based instructional progression (see Diagram 4 below) widely used in mathematics programs around the globe.

The three-step progression is based on Bruner's research demonstrating that learners learn better when they begin by representing a concept concretely, using objects, then move to representing this same concept using diagrams or pictures, before finally representing it abstractly, using symbols. This is particularly true of learners who struggle with mathematics (Jordan, Miller, and Mercer, 1998; Sousa 2008; Ketterlin-Geller, Chard, and Fien 2008, Flores, 2010).

Diagram 4. Bruner Three-Step Instructional Progression



13 For a summary of the CRA instructional progression, see Concrete, Representational, Abstract: Instructional Sequence, published by PaTTAN (Pennsylvania Training and Technical Assistance Network), available at https://www.pattan.net/CMSPages/GetAmazonFile.aspx?path=~\pattan\media\publications\2019%20accessible%20pdfs\cra-methods-3-9-20wba.pdf&hash=4ee5add68a50b35ac1279a143a522196d018c407445282aea1244a31c1039575&ext=.pdf

The CRA instructional progression is a gradual, systematic instructional approach proven to enhance learners' mathematics performance, particularly for struggling learners. Each stage builds on the previous stage, and learners move through the steps at varying rates. Some learners need to spend very little time at the concrete step before moving on to the representational or symbolic steps. In contrast, some may bypass the first two phases and move directly to symbolic representation (Bruner, 1966). These learners should not be compelled to always start with concrete representation. Others, however, will need to spend much more time exploring concepts using objects before moving on. Teachers confident with CRA use it to differentiate on-the-spot as they receive feedback from learners during instruction; this allows them to be more inclusive — targeted and explicit — as needed.

Each of the three steps in the CRA instructional progression is described below:

Step 1: A concrete (*learning through doing, movement, and action*) step, where learners represent or explore a new concept by manipulating objects. This can involve having learners use counters to represent "three" of something or to solve an addition problem.

Step 2: A representational (*learning through seeing or pictures*) stage, where learners use images or pictures to represent the new concept. For example, representing "three" by drawing three circles or interpreting a drawing as representing three counters and two counters.

Step 3: An abstract (*learning through abstract symbol*) stage, where learners represent the new math concept using abstract symbols, for example, representing a group of three objects or a drawing of three objects by the symbol "3."

Implementing the CRA instructional progression

- Begin by having learners represent concept using manipulatives (objects).
- Allow lots of time for them to represent a concept using different objects.
- Once learners are able to do so accurately and easily, represent the concept with pictures. Model concept for them.
- Allow lots of time for learners to represent concept using different pictures.
- Check understanding. When learners have understood, move to next step.
- Model for learners how to represent concept using symbols.
- Allow lots of time for learners to represent concept using symbols.
- If learners are struggling, move back to pictures and objects.
- Do not rush through steps.
- Recognize that learners will move through the steps at different rates.

Adapted from Research-based Education Strategies and Methods https://makingeducationfun.wordpress.com/2012/04/29/concrete-representational-abstract-cra/

ASSOCIATED PRACTICE

TRAINING: Train teachers on how to select appropriate manipulatives for key concepts and use them correctly, following the three-step CRA instructional model.

The presence of manipulatives, diagrams or other teaching and learning aids in a classroom does not guarantee improved learning (Baroody, 1989; Ball, 1992). The manipulatives or diagrams used must accurately represent the targeted mathematics concept. We encourage education technicians to design teacher training programs that provide guidance and practice on using materials or diagrams to enhance learning for key concepts at each grade level. Facilitators can show teachers how to use these materials or diagrams judiciously and intentionally and ensure that manipulatives are in the hands of learners (not the teacher) who can use them to both explore new concepts and represent their understanding.

Training programs can ensure that teachers are using the materials effectively and efficiently, that teachers move learners back and forth between representing a concept concretely, using pictures, and finally using symbols. This conscious moving back and forth between physical materials and picture and then symbolic representations gives meaning to abstract concepts. However, teachers need training on how to make these connections.

However, teachers may not use manipulatives if doing so increases class management problems. Teacher training programs need to focus on simple strategies for storing, distributing and collecting manipulatives in the classroom, and for ensuring learners are on-task during learning activities. The textbox provides guidance on how to select appropriate manipulatives for a given concept, as well as what to consider when planning a manipulative-based learning activity.

Checklist for trainers, coaches, and teachers

- Ensure the manipulatives accurately represent the mathematical concept.
- Ensure there are enough sets or utilize a routine sharing procedure so all learners can have manipulatives in their hands and can use them to explore or represent concepts (hands-on learning).

When planning activities with manipulatives:

- Ensure learners use them as a “thinking tool” to represent and talk about a math concept.
- Avoid activities that focus on copying how the teacher or other pupils use manipulatives.
- Ask learners to use manipulatives to help them explain or justify their thinking.

Adapted from Ontario Ministry of Education 2003, p. 20)

MATERIALS: Embed manipulatives and models — including those which are locally accessible — in model lessons and show teachers how to use to represent key concepts.

We encourage education technicians to design instructional materials that demonstrate to teachers how to use mathematical models (manipulatives, diagrams, etc.) judiciously and in a variety of ways, using low or no-cost materials sourced from the local environment whenever possible. These materials should also present accurate physical and pictorial representations of key concepts.

As part of the instructional materials development process, effective education technicians plan for local capacity building to develop and deliver manipulatives or seek alternative sustainable solutions for manipulatives that cannot be locally sourced. Solutions for local fabrication include private sector engagement through in-kind or sponsoring procurement of items. While many countries are keen to use locally available resources, commonly found objects in natural or recycled materials, other countries are adamant that learning materials meet key specifications.

Importance of using accurate physical models

When young learners are first introduced to place value, they need physical models that clearly show the relationship between units and groups of 10. This can be done by having loose sticks represent the numbers from one to nine, and having learners bundle up 10 loose sticks to make a group of 10. Physical models help learners clearly understand the difference, quantitatively, between one group of 10 sticks and one stick.

On the other hand, learners unfamiliar with place value presented with red and blue sticks and told that a red stick represents 10 and a blue stick represents one are likely to find the concept of place value confusing.

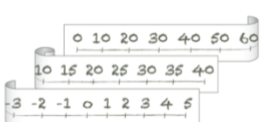
Education technicians should create teachers’ guides containing model lessons providing step-by-step guidance that clearly indicates how to use materials and diagrams effectively. The sequence of lessons in the guides should follow the CRA instructional model enabling teachers to gradually move learners’ thinking from concrete to abstract understanding.

COACHING: Support teachers to use manipulatives and diagrams accurately and effectively in their daily lessons.

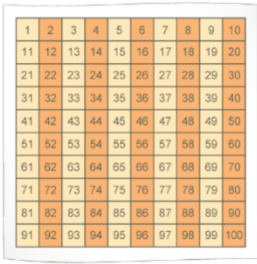
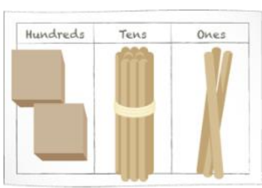
Opportunities for coaches to support teachers in judiciously choosing manipulatives, diagrams or other teaching and learning aids include professional development activities such as communities of learning sessions, model lessons, reviewing lesson plans in advance of teaching, or observation feedback cycles. Classroom observations should emphasize why and when to use manipulatives and models, and care must be taken so that coaches do not apply a checklist approach that can simply reinforce performative use or misuse of those materials.

Table 1 below provides an overview of common manipulatives and the mathematical concepts they can accurately represent.

Table 1: Common manipulatives in low-resource environments and the concepts they can represent[14]

Supplemental Material	Description	Mathematical concepts
Manipulative: Counters 	Single blocks, bottle caps, or other small objects used to represent single digit numbers, or at most numbers to 20	• Counting, representing single digit numbers or numbers to 20 • Skip counting • Basic addition, subtraction, multiplication and division facts • Equality • Decomposing numbers/number bonds
Manipulative: Base 10 blocks (or bundles) 	Base 10 blocks comprised of single cubes representing ones or units, long sticks of 10 cubes or units that represent 10, flats or squares made up of 10 long sticks of 10 cubes representing 100, and a large cube representing 100 made up of 10 flats or squares Base 10 bundles are made up of single sticks or straws representing ones or units, bundles of 10 sticks or units representing 10, and larger bundles of 10 smaller bundles of 10, representing 100	• Place value • Reading and representing numbers • Addition and subtraction of two- and three-digit numbers • Multiplication and division • Decimals and operations on decimals • Percentages • Expanded numeration
Anchor Chart: Number line 	A line with numbers in increasing order. Number lines can start at 0 or 1 and increase incrementally by 1 (or 2 or 3 or 10…)	• Counting • Skip counting • Increasing or decreasing number patterns • Addition and subtraction by counting up or down, or by adding or subtracting in "jumps" • Place value • Number relationships/number sense • Relative size of numbers

[14] Adapted from Concrete-Representational-Abstract Instructional Approach, Strategies for Teaching Elementary Mathematics: https://mathteachingstrategies.wordpress.com/2008/11/24/concrete-and-abstract-representations-using-mathematical-tools/ Accessed Dec. 23, 2020.

Supplemental Material	Description	Mathematical concepts
Anchor Chart: Hundreds chart 	A series of 100 squares in a 10 x 10 square grid, with numbers arranged sequentially in the squares from 0 to 100	• Counting • Skip counting • Increasing or decreasing number patterns • Addition and subtraction by counting up or down, or by adding or subtracting in "jumps" • Place value • Number relationships/number sense • Relative size of numbers • Percentages • Decimals
Manipulative: 5 Frame 	A rectangle divided into five equal squares, used in conjunction with counters	• Representing numbers to five • Decomposing numbers to five • Number bonds to five • Addition and subtraction to five
Manipulative: 10 Frame 	A rectangle divided into two rows of five equal squares, used in conjunction with counters	• Representing numbers to 20 or beyond • Decomposing numbers to 20 or beyond • Number bonds to 20 or beyond • Addition and subtraction to 20 or beyond
Manipulative: Place value mat 	A mat or piece of paper divided into three columns, labels "Hundreds," "Tens," and "Ones," used in conjunction with base 10 blocks or bundles above	• Place value • Decomposing numbers into 100s, 10s and ones • Comparing and ordering numbers • Addition and subtraction, with and without regrouping

Due to the precise specifications of manipulatives, local fabrication may not be feasible or sustainable. However, in some contexts these items may be locally sourced or create an opportunity for income generation or private sector engagement. Below, we outline some examples for supplemental materials for geometry:

Table 2: Common manipulatives requiring precise fabrication

Supplemental Material	Description	Mathematical concepts
Manipulative: Cuisenaires rods, multifix or unifix cubes 	Wooden rods: length of shortest rod represents one; length of next shortest rod (different color) is twice that of shortest rod; length of third smallest rod (different color) is three times that of shortest rod, etc., until there are 10 rods in total, with the length of the longest rod being 10 times the length of the shortest rod	• Numbers (integers, fractions, decimals) • Measurement • Geometry
Manipulative: Pattern blocks 	Pattern blocks are wooden blocks of six different shapes and colors: equilateral triangles, squares, lozenges, trapezoids and hexagons, with interesting relationships between them	• Geometry (symmetry, congruence, angles, similarity, paving) • Numbers (fractions)
Manipulative: Geometric solids 	3-D shapes that can be printed, cut or fabricated: cylinders, cones, pyramids, cubes, cuboids, prisms, and spheres Found objects can sometimes work; PVC pipe, an orange, an eraser, soda can, cardboard box, ball, etc.	• Geometry (properties of solids) • Surface area • Volume

PRINCIPLE 8: Develop math fact fluency through reasoning and practice.

Practitioner's note: This principle can be generalized to address all elements of automaticity, defined as learners' ability to quickly identify or access information. This can involve being able to recognize and read numerals, identify two- or three-dimensional forms or figures, or access definitions, algorithms or formulas.

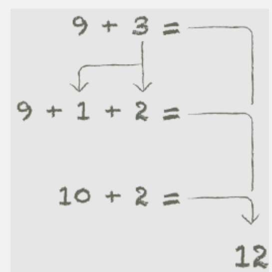
Developing automaticity or fluency in mathematics is as important as it is in reading. Learners who can quickly and accurately access basic information are able to devote more short-term memory to other aspects of problem solving. Not surprisingly, learners who have developed greater automaticity or fluency in accessing basic information score higher on math assessments. We believe that when designed well, activities that promote fact fluency can bring joy to learners through games and play that deliver a daily dose of challenge, suspense, drama, song, dance, humor, collaboration, and cooperation (See ***Every Learner Can be Numerate***).

Procedural versus computational fluency. The National Council of Teachers of Mathematics (NCTM) defines procedural fluency or automaticity as the “ability to apply procedures accurately, efficiently, and flexibly; to transfer procedures to different problems and contexts, to build or modify procedures from other procedures; and to recognize when one strategy is more appropriate to apply than another” (NCTM 2014, quoted in Evans et al., 2018). Computational *fluency,* on the other hand, is the ability to perform operations on numbers quickly and accurately. Up to 80 percent of the mathematical computations performed in non-technical settings, such as the exchange of money or the determination of times and distances, are done mentally (Reys and Nohda, 1994, quoted in Evans et al., 2018).

Importance of computational fluency. Two decades of research have established a correlation between fluency or automaticity with number facts and better mathematics performance (Fuchs, Seethaler, et al., 2008; Fuchs, et al., 2006; Fuchs et al. 2005; Fuchs, Powell, et al., 2008; Tournaki, 2003; Woodward, 2006). Learners who can retrieve number facts accurately and quickly are better able to focus their cognitive resources (i.e., short term memory) on other aspects of a problem.

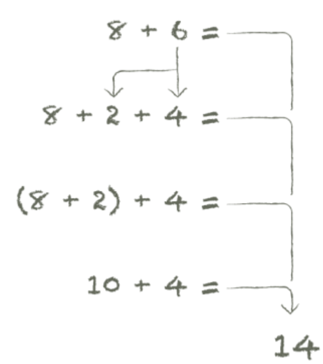
Reason-based strategies to develop computational fluency

Making “friendlier” number – i.e., decomposing a number to make numbers that are “friendlier” or easier to deal with, for example, changing 9 + 3 to 9 + 1 + 2 and subsequently 10+ 2 by shift “1” from the 3 to the 9 to create a 10.



How to develop learners’ computational fluency. Although educators generally agree on the importance of developing learners’ computational fluency, there is some debate on how to best achieve that goal. Research carried out by Gray and Tall (1994) and Boaler and Zoido (2016) found that high performing learners tended to use computational strategies grounded in a conceptual understanding of the number system. Lower-achieving learners, however, tended to rely on memorization, a far less effective and efficient strategy (ibid).[15]

Some researchers maintain that the basis of computation fluency is in fact a strong conceptual understanding of the number system — not memorization (Boaler, 2009, 2015, 2016).[16] Learners begin developing this understanding when they use manipulatives to explore different ways that quantities can be composed and decomposed (see Principle 7); for example, that 6 can be represented as 4 and 2, 1 and 5, or 3 and 3. Learners who can represent quantities in equivalent ways can leverage that knowledge to develop computational fluency. For example, when faced with a difficult addition problem (see example below), they can replace the 6 by an equivalent addition (2 + 4) to create an equivalent addition that is easier to solve (Evans et al., 2018).



Researchers like Boaler (2009, 2015, 2016) or Gray and Tall maintain that the best way to develop fluency with numbers is to develop learners’ number sense so they are able to work with numbers in

[16] For discussion of the role of fluency in mathematics and research-based methods for developing fluency in young learners, see Boaler, J. (2015). Fluency Without Fear: Research Evidence on the Best Ways to Learn Math Facts . https://www.youcubed.org/evidence/fluency-without-fear/ accessed January 2, 2021.

different ways, as opposed to blindly memorizing number facts.[17] They maintain that over-emphasis on memorization and timed drill and practice as a means of developing fluency, as opposed to number sense, can inhibit learners' mathematical development.

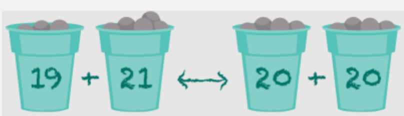
Role of practice in developing computational fluency. Although the basis of computational fluency is the development of number sense — defined as a strong conceptual understanding of the number system — learners need considerable practice applying number-sense-based strategies if they are to develop automaticity or fluency with basic number facts. Simple number games are likely to be familiar to teachers and can be easily integrated into existing practices. They are also a source of enjoyment for learners and can motivate to develop fluency with number facts. Games or other activities enable learners to quickly identify an appropriate strategy for a given computation and apply it accurately (see below).

ASSOCIATED PRACTICE

TRAINING: Train teachers how to implement number talks or other activities to develop learners' fluency with number facts and relationships.

Sample number talk activities

- I will say a number. You need to say the number that, when added to this number, gives 10.
- I will say a number. You need to say the double of the number.
- Use the strategy "making friendly numbers" to add 19 + 21.



- Use the strategy "making doubles" to add the numbers like 6 + 5 or 49 + 52 or 7 + 8.
- Use the strategy "Subtracting in chunks" and jump backward on a number line or hundreds chart to subtract numbers like 25 – 12.

Pre-service or in-service training programs should introduce teachers to the importance of developing fluency or automaticity with basic number facts and relationships, and evidence-based strategies for doing so at each grade level. This includes providing teachers with hands-on practice during trainings on carrying out number talk games, songs, and activities so they are able to do so quickly, accurately and with confidence.

MATERIALS: Teachers' guides or other instructional materials should be carefully sequenced with grade-level-appropriate activities for the implementation of daily, five-minute "number talks"[18] to develop fluency or automaticity through mathematical reasoning.

Education practitioners should advocate that teachers be provided with games and activities — aligned with the learning progressions and the curriculum — and encourage teachers to schedule daily, five-minute "talks" as a short activity at the beginning of lessons. Teachers can pose an abstract math problem such as 18 x 5 and ask learners to solve it mentally. The teacher then collects the different methods used (for example multiplying 20 x 5 to get 100, then subtracting two groups of five (or 10) from the answer or multiplying 10

[17] PISA testing is a form of international testing in mathematics and science given to 15-year-olds worldwide. In 2012, the Organization for Economic Co-Operation and Development (OECD) collected evidence of students' approaches to learning and focused on mathematics in particular. Analysts divided students into one of three learning styles: students either approached mathematics learning by memorizing, by relating material to previously known material, or by self-monitoring — connecting new ideas to those they had learned. Analysts found that students who took a memorization approach to mathematics were the lowest achieving students in every country and any country that had high numbers of memorizers (such as the US) was low performing. (Boaler 2019, Limitless Mind: Learn, Lead & Live without Barriers. (Harper Collins 2019).

[18] For more about implementing number talks in the classroom, see Humphreys, Cathy, and Ruth Parker. 2015. Making Number Talks Matter: Developing Mathematical Practices and Deepening Understanding, Grades 4 – 10. Portland, ME: Stenhouse.

x 5 and then 8 X 5 and adding the answers to the two multiplications), and learners can then examine why each method works.[19]

Number talk activities can also focus on the development of fluency or automaticity through games and other engaging activities (see examples in text box). In pre-primary and early primary classrooms, these activities begin with simple counting activities to develop fluency with counting, and gradually progress to the exploration and use of reasoning strategies for adding and subtracting (counting up, counting down, skip counting by 10s or 5s, using fingers to identify how much more it takes to make five or 10, decomposing a number to make combinations that are easier to add or subtract), and eventually to multiplying and dividing. In upper primary, number talks focus on developing fluency with number operations, but using larger numbers (reasoning strategies for adding or subtracting two- or three-digit numbers, or for multiplying or dividing larger numbers).

COACHING: Support teachers to implement effective fluency-building activities in daily lessons.

Simple tools to collect data help reveal whether teachers are including fluency-building activities in their lessons, and how effective these activities are. There are two important questions for teachers and coaches to consider: (1) Does the fluency-building activity reinforce previously taught mathematical concepts, and (2) Is the fluency-building activity inclusive, so that learners struggling with confidence or conceptual gaps are not further excluded or experience emotional disassociations? It is also important to support teachers in using fluency-building activities to celebrate “building,” and practice a growth mindset for focusing on how many more correct facts did learners get today than yesterday, or this week than last week, rather than focusing only on how many facts learners got wrong. Communities of learning can problem-solve around fluency-building activities and strategize collaborative ways to make them meaningful and joyful for all learners.

Pulling it altogether — Making mathematics meaningful

Not relying on rote teaching activities and testing memorized math facts is always a challenge, especially in overcrowded and/or under-resourced classrooms. However, active learning, exploration, and fostering a growth mindset are tools for effective numeracy programming. The (2+6) research-based principles and practices can make mathematics meaningful for learners. When learning is meaningful, learners gain a deeper understanding of the mathematics involved (Grossman, 2012).

Using open-ended questions to make mathematics more meaningful

Open-ended questions like “What did you notice about…” or “How might we…” make mathematics more meaningful. They are not designed to illicit a single, correct response, but rather to engage learners in open-ended conversations that promote higher-level thinking.

The (2+6) principles are designed around the idea that learners at all grade levels are natural mathematicians: they naturally push, pull, build, stack, measure, count, share, and group. They recognize, identify, build, and extend patterns. They compare and observe what is the same and what is different, classifying shapes and objects according to these observations. They describe and make decisions based on informal calculations of probability: Should I wear a sweater today? What are the odds that the teacher would check to see if I completed my homework?

[19] See Boaler, 2015 for other examples of number talk activities.

In the following chapters we will explore practical techniques and evidence-based practices that education technicians can use to improve instruction quality and learning outcomes.

Diagram 5. Changes That Make Mathematics More Meaningful for Learners

Less meaningful	More meaningful
Focus only on formal mathematics in curriculum and textbook	Link formal mathematics concepts to their informal application in the classroom
	Link formal mathematics to applications in learners' out-of-school interests and passions
Use routine problems	Modify textbook problems to be more open-ended and challenging
	Modify textbook problems to align with learners' interest or topics of interest to local community
	Create problems that reflect learners' interests or passions
Tell or explain a concept or a procedure	Give learners a problem that builds an understanding of a concept or procedure (learner-led problem solving)
	Have learners explain or tell
Show using diagrams or manipulatives	Have learners show and explain, using diagrams or manipulatives
Drill basic number facts to improve recall, memorization	Teach learners reasoning strategies for solving basic number facts; use games to develop their fluency with these strategies

Table 3 summarizes the (2+6) Guiding Principles of effective numeracy instruction and associated practices that support teachers and learners in the math classroom.

Table 3: Summary of the (2+6) Guiding Principles of effective numeracy instruction and associated practices

Principle	Associated Practices		
	Training	TLMs	Support/Coaching
Two Foundational Guiding Principles			
Create inclusive classroom environments that foster math-positivity, perseverance, and risk-taking.	Raise teacher awareness that everyone can learn and become numerate. Introduce simple and effective teaching techniques that are inclusive and enable a math-positive environment.	Provide teachers with tools and opportunities to practice and hone teacher craft with simple, inclusive techniques. Whenever possible, link to distance learning and technology platforms where students can practice.	Support teachers for daily classroom use of effective techniques judiciously and gradually. Align support from coaches and communities of learning with principles of structured pedagogy, and math-positive teaching approaches. Reinforce this support in training and coaching tools.

Principle	Associated Practices		
	Training	TLMs	Support/Coaching
Teach responsively based on evidence. Assess regularly to identify and fill learning gaps.	Train teachers on simple formative assessment practices, including how to use simple tools to monitor progress against milestones. Integrate effective teaching techniques so teachers are empowered on "how" to respond to data from real time interactions from the "what" provided by teachers' guides, textbooks, and supplemental materials.	Ensure model lessons include five to 10 minutes of independent practice and activities to assess learners' progress and address identified learning gaps. Provide teachers with simple tools to measure learner progress against milestones, and simple, follow-up differentiated learning activities based on results. Where possible, introduce technology-based solutions to support assessment.	Support teacher's use of simple formative assessment practices, understanding of feedback. Institute follow-up with differentiated learning opportunities. Support instructional coaches to facilitate reflections on evidence of student learning and responsive decision making in dialogues with teachers.
Six Numeracy- and Mathematics-Specific Guiding Principles			
Respect learning progressions. Structure instruction according to research-based learning progressions. Pace learning to allow for development of deep understanding.	Train teachers on developmental progressions and structured pedagogy to support key skills. Ensure instructional materials are structured according to math learning progressions.	Develop carefully sequenced math curricula and/or instructional materials that reflect research-based developmental learning progressions.	Support teacher's use in the classroom of evidence-based instructional materials based on learning progressions and associated instructional practices.
Connect formal mathematics learned in school with informal mathematics used outside of school.	Raise teacher awareness of importance of connecting informal and formal mathematics and provide them with strategies to do so.	Ensure new mathematics instructional materials link math concepts to how these concepts are used by learners and community members outside the classroom.	Provide teachers with in-class support on how to build bridges between informal and formal mathematics in the classroom.
Promote learning through non-routine problem-solving.	Raise teacher awareness of the power of non-routine problems, the importance of a learner-led problem solving, and of key activities for each stage of learner-led problem-solving lessons.	Ensure new instructional materials include models of learner-led, problem-solving lessons and the construction, deconstruction, reconstruction, and instructional progression.	Support teachers to implement learner-led problem-solving model lessons.

Principle	**Associated Practices**		
	Training	TLMs	Support/Coaching
Encourage learner explanation and justification of mathematical reasoning.	Train teachers on the value of learner “talk” and the simple, generic questions that encourage students to explore and explain in the classroom.	Provide teachers with instructional materials that model how to facilitate learner-led talk in the classroom.	Support teachers to increase learner-led talk in the classroom by setting up appropriate activities and asking questions that promote explanation and justification.
Use manipulatives, tools, and models intentionally.	Train teachers on how to accurately and intentionally select appropriate manipulatives for key concepts and use them correctly, following the three-step CRA instructional model.	Embed learning activities or model lessons in instructional materials showing teachers how to use locally available manipulatives, tools, and models to represent key concepts.	Support teacher’s use of manipulatives and diagrams accurately and effectively in their daily lessons, following the three-step CRA instructional model.
Develop learner fluency with number facts	Train teachers to build math fact fluency through exercises and games rather than rote memorization.	Ensure teachers’ guide or other instructional materials include carefully sequenced and grade-level appropriate activities.	Support teachers to implement effective fluency-building activities in their daily lessons.